

# Note on Minkowski's Determinant Inequality

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## 1 Statement of the Problem

The present note is concerned with the following theorem.

**Theorem 1.1.** *Let  $A, B \in \mathbb{C}^{n \times n}$  be Hermitian and positive semi-definite. We have the next inequality.*

$$(\det(A + B))^{1/n} \geq (\det A)^{1/n} + (\det B)^{1/n}. \quad (1)$$

The proof demonstrated below might not be straightforward. While we can probably simplify it, the proof is more or less attractive because we encounter a lot of interesting inequalities.

It might be important to note that the Minkowski's determinant inequality can be considered as a special case of a more general inequality related to functions of positive definite matrices. Interested readers can refer to [1].

## 2 Proof of the Theorem

We prove the theorem in several steps.

### 2.1 First Weapon

In this subsection, we prove the following result.

**Proposition 2.1.** *For  $a_1, \dots, a_n, b_1, \dots, b_n \geq 0$ , it holds that*

$$\prod_{j=1}^n (a_j + b_j)^{1/n} \geq \prod_{j=1}^n a_j^{1/n} + \prod_{j=1}^n b_j^{1/n}.$$

To begin with, we review a famous inequality.

**Lemma 2.2.** *Let  $a_1, \dots, a_n \geq 0$ . It holds that*

$$\prod_{j=1}^n a_j^{1/n} \leq \frac{1}{n} \sum_{j=1}^n a_j.$$

*Proof.* If some of  $a_j$  are 0, the inequality becomes trivial. Let us assume  $a_j > 0$  for every  $j = 1, \dots, n$ . We adopt the proof that makes use of Jensen's inequality. Since log is concave,

$$\log \left[ \prod_{j=1}^n a_j^{1/n} \right] = \sum_{j=1}^n \frac{\log a_j}{n} \leq \log \left[ \frac{1}{n} \sum_{j=1}^n a_j \right],$$

by Jensen's inequality. The monotonicity of log immediately concludes the desired result. ///

Concerning geometric means, we have the next inequality due to Hoehn and Niven [2].

**Lemma 2.3 (Hoehn and Niven [2]).** *Suppose  $a_1, \dots, a_n > 0$ . For any  $c > 0$ , it holds that*

$$\prod_{j=1}^n (c + a_j)^{1/n} \geq c + \prod_{j=1}^n a_j^{1/n}.$$

*Proof.* Let us define the function

$$G(x) := \prod_{j=1}^n (x + a_j)^{1/n}, \quad x \geq 0.$$

The proof boils down to showing the derivative function satisfies  $G'(x) \geq 1$ . Suppose this is true. By the mean value theorem, there is some  $x \in (0, c)$  that satisfies

$$\frac{G(c) - G(0)}{c - 0} = G'(x) \geq 1.$$

Hence,  $G(c) \geq c + G(0)$ , which is exactly what we wanted.

To see the derivative condition  $G'(x) \geq 1$ , first note that

$$\frac{d}{dx} G(x)^n = nG(x)^{n-1} G'(x) = \sum_{i=1}^n \prod_{j \neq i} (x + a_j).$$

It, therefore, suffices to show that

$$G(x)^{n-1} \leq \frac{1}{n} \sum_{i=1}^n \prod_{j \neq i} (x + a_j).$$

This is immediate from Lemma 2.2, since

$$G(x)^{n-1} = \prod_{j=1}^n \prod_{j \neq i} (x + a_j)^{1/n}.$$

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Now we are ready to prove Proposition 2.1.

*Proof of Proposition 2.1.* If some of  $a_j, b_j, j = 1, \dots, n$ , are 0, the target inequality is trivial. For the non-trivial case apply Lemma 2.3 to obtain

$$\prod_{j=1}^n \left(1 + \frac{b_j}{a_j}\right)^{1/n} > 1 + \prod_{j=1}^n \left(\frac{b_j}{a_j}\right)^{1/n}. \quad (2)$$

By multiplying  $\prod_{j=1}^n a_j^{1/n}$  to both sides of (2), we obtain the desired result. ///

## 2.2 Second Weapon

The next result we will prove here is summarized as follows.

**Proposition 2.4.** *Let  $A \in \mathbb{C}^{n \times n}$  be Hermitian and nonnegative definite. Let  $\{v_1, \dots, v_n\}$  be an arbitrary orthonormal basis of  $\mathbb{C}^n$ . The following inequality is true.*

$$\prod_{j=1}^n v_j^* A v_j \geq \det A.$$

This Proposition follows from the next Lemma.

**Lemma 2.5 (Hadamard's Inequality).** *Let  $A = [a_{ij}] \in \mathbb{C}^{n \times n}$  be Hermitian and nonnegative definite. Then, the next inequality holds.*

$$\prod_{j=1}^n a_{jj} \geq \det A.$$

*Proof.* Let  $[e_1 \dots e_n]$  be the natural basis<sup>1</sup> and  $[v_1 \dots v_n]$  the orthonormal basis that consists of eigenvectors for

<sup>1</sup>That is,  $a_{ij} = e_i^* A e_j$ .

A. Corresponding eigenvalues are denoted by  $\lambda_j, j = 1, \dots, n$ . Define  $V := [v_1 \dots v_n]$ . Since  $V$  is unitary, by

$$[v_1 \dots v_n] = [e_1 \dots e_n]V,$$

we have  $e_j = \sum_{k=1}^n V_{kj}^* v_k = \sum_{k=1}^n \bar{v}_{jk} v_k$ . Hence,

$$\begin{aligned} \prod_{j=1}^n a_{jj} &= \prod_{j=1}^n e_j^* A e_j \\ &= \prod_{j=1}^n \left[ \left( \sum_{k=1}^n \bar{v}_{jk} v_k \right)^* A \left( \sum_{l=1}^n \bar{v}_{jl} v_l \right) \right] \\ &= \prod_{j=1}^n \left[ \sum_{k=1}^n \sum_{l=1}^n v_{jk} \bar{v}_{jl} v_k^* A v_l \right] \\ &= \prod_{j=1}^n \left[ \sum_{k=1}^n |v_{jk}|^2 \lambda_k \right] \\ &\geq \prod_{j=1}^n \prod_{k=1}^n \lambda_k^{|v_{jk}|^2} \\ &= \prod_{k=1}^n \lambda_k^{\sum_{j=1}^n |v_{jk}|^2} \\ &= \prod_{k=1}^n \lambda_k = \det A. \end{aligned}$$

The inequality in the fifth line is the generalized version of the inequality concerning arithmetic mean and geometric mean, which can be proved in much the same way as Lemma 2.2. ///

*Proof of Proposition 2.4.* Suppose that  $\{v_1, \dots, v_n\}$  is orthonormal.  $V := [v_1 \dots v_n]$  is unitary and  $v_j^* A v_j$  is  $j$ -th diagonal element of the matrix  $V^* A V$ . By Hadamard's inequality, we obtain

$$\prod_{j=1}^n v_j^* A v_j \geq \det V^* A V = \det A,$$

as was desired. ///

## 2.3 Proof of Theorem 1.1

Let  $\{v_1, \dots, v_n\}$  be the orthonormal system of eigenvectors for Hermitian matrix  $A + B$ .

$$\begin{aligned}
(\det(A + B))^{1/n} &= \left( \prod_{j=1}^n v_j^*(A + B)v_j \right)^{1/n} \\
&= \prod_{j=1}^n (v_j^*Av_j + v_j^*Bv_j)^{1/n} \\
&\geq \prod_{j=1}^n (v_j^*Av_j)^{1/n} + \prod_{j=1}^n (v_j^*Bv_j)^{1/n} \\
&= \left( \prod_{j=1}^n v_j^*Av_j \right)^{1/n} + \left( \prod_{j=1}^n v_j^*Bv_j \right)^{1/n} \\
&\geq (\det A)^{1/n} + (\det B)^{1/n}.
\end{aligned}$$

The first inequality is derived from Proposition 2.1, and the second from Proposition 2.4. ///

## References

- [1] Bhatia, R. (2006) *Positive Definite Matrices*: Princeton University Press.
- [2] Hoehn, L. and I. Niven (1985) "Averages on the Move," *Mathematics Magazine*, Vol. 58, pp. 151–156.