Notes on Hardy's Inequalities

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December 23, 2009

This note is principally based on Steele [1].

1 Integration Version

Theorem 1.1 (Hardy's Inequaliy (I)). *For every integrable function* $f : [0, T] \to \mathbb{R}$,

$$\int_0^T \left\{ \frac{1}{t} \int_0^t f(s) ds \right\}^2 dt \leq 4 \int_0^T f^2(t) dt. \tag{1}$$

We first prove the next lemma.

Lemma 1.2. *For every integrable function* $f : [0, T] \to \mathbb{R}$,

$$\int_0^T \left\{ \frac{1}{t} \int_0^t f(s) ds \right\}^2 dt \leq 2 \int_0^T \left\{ \frac{1}{t} \int_0^t f(s) ds \right\} f(t) dt. \tag{2}$$

Proof. Let's start from the left-hand side. Integration by parts formula gives us

$$\begin{aligned} &\int_0^T \left\{ \frac{1}{t} \int_0^t f(s) ds \right\}^2 dt = \int_0^T \frac{1}{t^2} \left\{ \int_0^t f(s) ds \right\}^2 dt \\ &= \left[-\frac{1}{t} \left\{ \int_0^t f(s) ds \right\}^2 \right]_0^T + \int_0^T \frac{1}{t} \left\{ 2f(t) \int_0^t f(s) ds \right\} dt \\ &= -\frac{1}{T} \left\{ \int_0^T f(t) dt \right\}^2 + 2 \int_0^T \left\{ \frac{1}{t} \int_0^t f(s) ds \right\} f(t) dt. \quad (3) \end{aligned}$$

The last equality holds because

$$\begin{aligned} 0 &\leq \lim_{t \searrow 0} \frac{1}{t} \left\{ \int_0^t f(s) ds \right\}^2 \leq \lim_{t \searrow 0} \frac{1}{t} \int_0^t ds \int_0^t f(s)^2 ds \\ &= \lim_{t \searrow 0} \int_0^t f(s)^2 ds = 0. \end{aligned}$$

Discarding the first term in the last line of (3), we obtain (2). ///

We are ready to prove the integration version of the Hardy's inequalities.

Proof of the Hardy's Inequality (I). For the sake of notational simplicity let's define $\phi(t) := \frac{1}{t} \int_0^t f(s) ds$ and $\psi(t) := f(t)$. (2) becomes

$$\int_0^T \phi^2(t) dt \leq 2 \int_0^T \phi(t) \psi(t) dt.$$

Applying the Cauchy–Schwarz inequality to the right-hand side we have

$$\int_0^T \phi^2(t) dt \leq 2 \left\{ \int_0^T \phi^2(t) dt \right\}^{1/2} \left\{ \int_0^T \psi^2(t) dt \right\}^{1/2}.$$

If $\phi = 0$, $f = 0$ and hence (1) is trivial. So, it is fair to assume $\phi \neq 0$. When this is the case, we readily derive

$$\int_0^T \phi^2(t) dt \leq 4 \int_0^T \psi^2(t) dt,$$

which is the same inequality as was given in (1) from the definition of ϕ and ψ. ///

2 Summation Version

Theorem 2.1 (Hardy's Inequaliy (II)). *For every real-valued sequence* $(a_n)_{n=1}^N$,

$$\sum_{n=1}^N \left(\frac{1}{n} \sum_{k=1}^n a_k \right)^2 \leq 4 \sum_{n=1}^N a_n^2. \tag{4}$$

Let's prove the next lemma.

Lemma 2.2. *For every* $N \in \mathbb{N}$ *and* $a_1, \ldots, a_n \in \mathbb{R}$,

$$\sum_{n=1}^N \left(\frac{1}{n} \sum_{k=1}^n a_k \right)^2 \leq 2 \sum_{n=1}^N \left(\frac{1}{n} \sum_{k=1}^n a_k \right) a_n. \tag{5}$$

Proof. Define $A_n := \frac{1}{n}\sum_{k=1}^{n} a_k$. Calculating difference of each summand,

$$
\begin{aligned}
&\left(\frac{1}{n}\sum_{k=1}^{n} a_k\right) a_n - \left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)^2 \\
&= A_n a_n - A_n^2 \\
&= a_n^2 + -(A_n - a_n)^2 \\
&= (2a_n - A_n)A_n \\
&= 2\left[A_n\left\{nA_n - (n-1)A_{n-1}\right\}\right] - A_n^2 \\
&= (2n-1)A_n^2 - 2(n-1)A_nA_{n-1} \\
&\geq (2n-1)A_n^2 - (n-1)(A_n^2 + A_{n-1}^2) \\
&= nA_n^2 - (n-1)A_{n-1}^2.
\end{aligned}
$$

Sum up the above inequality to obtain

$$
\sum_{n=1}^{N}\left[\left(\frac{1}{n}\sum_{k=1}^{n} a_k\right) a_n - \left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)^2\right] \geq NA_n^2 \geq 0.
$$

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Proof of the Hardy's Inequality (II). Application of the Schwarz inequality to the right-hand side of (5),

$$
\begin{aligned}
\sum_{n=1}^{N}\left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)^2 &\leq 2\sum_{n=1}^{N}\left(\frac{1}{n}\sum_{k=1}^{n} a_k\right) a_n \\
&\leq 2\sqrt{\sum_{n=1}^{N}\left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)^2}\sqrt{\sum_{n=1}^{N} a_n^2}.
\end{aligned}
$$

Thefore, unless a_n are identically zero,

$$
\begin{aligned}
\sqrt{\sum_{n=1}^{N}\left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)^2} &\leq 2\sqrt{\sum_{n=1}^{N} a_n^2}, \\
\therefore \quad \sum_{n=1}^{N}\left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)^2 &\leq 4\sum_{n=1}^{N} a_n^2.
\end{aligned}
$$

If a_n are identically zero is trivial the target inequality (4) is trivial.

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References

[1] Steele, J. M. (2004) *The Cauchy–Schwarz Master Class*: Cambridge University Press.