Advanced Microeconomics: Homework 2

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5 Define the preference relation u on $\mathbb{R}^2_{++}$ as

$$u(x) := \frac{1}{1-\alpha}x_1^{1-\alpha} + \frac{1}{1-2\alpha}x_2^{1-2\alpha},$$

where $\alpha > 1$ and $x = (x_1, x_2)$. Let $x(p, w)$ denote the demand function of the preference relation that u represents. For a given (p, w),

$$\frac{w}{x_l(p,w)}\frac{\partial x_l}{\partial w}(p,w)$$

is said to be the income elasticity of demand of the l-th commodity ($l = 1, 2$) at (p, w). Prove that the income elasticity of demand of the 1st commodity is larger than 1 and that of the 2nd commodity is less than 1.

Proof. We now consider the utility maximization problem:

$$\max_x u(x) \quad \text{subject to } p \cdot x \leq w.$$

The first order conditions for the optimality of this problem are

$$x_1^{-\alpha} - p_1\lambda = 0, \tag{1}$$

$$x_2^{-2\alpha} - p_2\lambda = 0, \tag{2}$$

$$p_1x_1 + p_2x_2 = w, \tag{3}$$

where $\lambda > 0$ is the Lagrangian multiplier. These are also sufficient for the optimality because u defines strictly convex preference. From (1) and (2) we have

$$p_1x_1^{\alpha} = p_2x_2^{2\alpha}. \tag{4}$$

Differentiating (4) with respect to w gives us

$$\frac{1}{x_1}\frac{\partial x_1}{\partial w} = \frac{2}{x_2}\frac{\partial x_2}{\partial w}.$$

Thus we get

$$e_1 = 2e_2, \tag{5}$$

where $e_l := (w/x_l)(\partial x_l/\partial w)$, $l = 1, 2$.
By differentiating both sides of (3), we obtain

$$p_1 \frac{\partial x_1}{\partial w} + p_2 \frac{\partial x_2}{\partial w} = 1. \tag{6}$$

From the identity (6), we have

$$\begin{aligned} \frac{w}{x_1}\frac{\partial x_1}{\partial w} + \frac{p_2 w}{p_1 x_1}\frac{\partial x_2}{\partial w} &= \frac{w}{p_1 x_1}, \\ \therefore e_1 + \frac{p_2 x_2}{p_1 x_1} e_2 &= \frac{w}{p_1 x_1}. \end{aligned} \tag{7}$$

The identities (5) and (7) get us

$$\left(2 + \frac{p_2 x_2}{p_1 x_1}\right) e_2 = \frac{w}{p_1 x_1}.$$

Noting, by (3), that $p_1 x_1 + p_2 x_2 = w$,

$$e_2 = \frac{w}{p_1 x_1} \frac{p_1 x_1}{2p_1 x_1 + p_2 x_2} = \frac{w}{p_1 x_1 + w} = 1 - \frac{p_1 x_1}{p_1 x_1 + w}. \tag{8}$$

For the first commodity, we have

$$e_1 = 2e_2 = \frac{2w}{p_1 x_1 + w} = 1 + \frac{p_2 x_2}{p_1 x_1 + w}. \tag{9}$$

Now it is clear by (9) and (8) that $e_1 > 1$ and $e_2 < 1$. ///