Linear Homogeneity + Quasi-Concavity $\Rightarrow$ Concavity *

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1 Problem

Suppose $f : \mathbb{R}^n_+ \to \mathbb{R}_+$ meets the following assumptions.

Assumption 1.1 (No Free Lunch). There is no possibility of free lunch: $f(0) = 0$.

Assumption 1.2 (Monotonicity). Function f is monotone increasing: For any $x_1,\ x_2 \in \mathbb{R}^n$ satisfying $x_1 \geq x_2$, $f(x_1) \geq f(x_2)$ holds.

Assumption 1.3 (Quasi-Concavity). Function f is quasi-concave: For any $x_1,\ x_2 \in \mathbb{R}^n_+$ and $t \in (0, 1)$, $f(tx_1 + (1-t)x_2) \geq \min\{f(x_1),\ f(x_2)\}$.

Assumption 1.4 (Linear Homogeneity). Function f is linearly homogeneous: For any $\lambda > 0$ and $x \in \mathbb{R}^n_+$, $f(\lambda x) = \lambda f(x)$.

Under these assumptions we have cocavity of f.

Problem 1.5. *Prove that f is concave: For any $x_1,\ x_2 \in \mathbb{R}^n_+$ and $t \in (0, 1)$, $f(tx_1 + (1-t)x_2) \geq tf(x_1) + (1-t)f(x_2)$).*

2 Proof

Without loss of generality we may assume $f(x_1) \leq f(x_2)$. We first assume that $f(x_1) > 0$ and that x_1 and x_2 are linearly independent. Note that there are $\lambda_1, \lambda_2 > 0$ such that

$$f(x_1) = f(\lambda_2 x_2), \quad f(x_2) = f(\lambda_1 x_1).$$

Define $\bar{x}_1 = \lambda_1 x_1,\ \bar{x}_2 := \lambda_2 x_2$. The above identities become

$$f(x_1) = f(\bar{x}_2), \quad f(x_2) = f(\bar{x}_1).$$

Claim 2.1. *Vectors $x_1 - \bar{x}_2$ and $\bar{x}_1 - x_2$ are parallel to each other.*

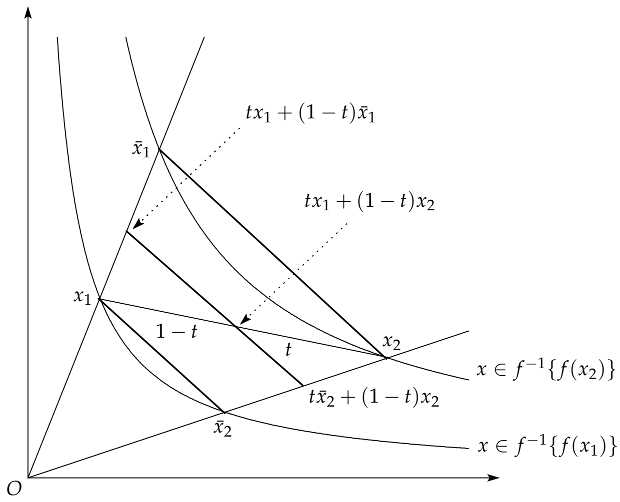

Figure 1: Projection onto the plane spanned by x_1 and x_2.

Proof. By linear homogeneity of f, $f(x_1) = \lambda_2 f(x_2)$ and $f(x_2) = \lambda_1 f(x_1)$ hold. We therefore have $\lambda_1\lambda_2 = 1$. Then,

$$\begin{aligned} x_1 - \bar{x}_2 &= \lambda_1^{-1}\bar{x}_1 - \lambda_2 x_2 \\ &= \lambda_1^{-1}(\bar{x}_1 - \lambda_1\lambda_2 x_2) \\ &= \lambda_1^{-1}(\bar{x}_1 - x_2). \end{aligned}$$

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Claim 2.2. *Point $tx_1 + (1-t)x_2$ is on the line segment between $tx_1 + (1-t)\bar{x}_1$ and $t\bar{x}_2 + (1-t)x_2$.*

Proof. Note that the line segment between $tx_1 + (1-t)\bar{x}_1$ and $t\bar{x}_2 + (1-t)x_2$ is parallel to that be-

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tween $\bar{x}_1$ and x_2. To see this,

$$
\begin{aligned}
&(tx_1 + (1-t)\bar{x}_1) - (t\bar{x}_2 + (1-t)x_2) \\
&= (t\lambda_1^{-1} + (1-t))\bar{x}_1 - (t\lambda_2 + (1-t))x_2 \\
&= \lambda_1^{-1}\left((t + \lambda_1(1-t))\bar{x}_1 - (t\lambda_1\lambda_2 + \lambda_1(1-t))x_2\right) \\
&= \lambda_1^{-1}(t + \lambda_1(1-t))(\bar{x}_1 - x_2).
\end{aligned}
$$

Elementary geometry will completes the proof. ///

Theorem 2.3. *Let f satisfy all the assumptions given above. For any x_1, $x_2 \in \mathbb{R}^n$ and $t \in (0,1)$, it holds that*

$$f(tx_1 + (1-t)x_2) \geq tf(x_1) + (1-t)f(x_2).$$

Proof. Without loss of generality, we may assume $f(x_1) \leq f(x_2)$. If x_1 and x_2 are linearly dependent, the target inequality holds true with equality by linear homogeneity. Hereafter, we assume otherwise.

We first consider the case in which $f(x_1) > 0$. After simple calculation we obtain

$$
\begin{aligned}
f(tx_1 + (1-t)\bar{x}_1) &= f(t\bar{x}_2 + (1-t)x_2) \\
&= tf(x_1) + (1-t)f(x_2).
\end{aligned}
$$

Since, $tx_1 + (1-t)x_2$ is on the line segment between $tx_1 + (1-t)\bar{x}_1$ and $t\bar{x}_2 + (1-t)x_2$, we obtain, by quasi-concavity,

$$
\begin{aligned}
&f(tx_1 + (1-t)x_2) \\
&\geq \min\{f(tx_1 + (1-t)\bar{x}_1),\ f(t\bar{x}_2 + (1-t)x_2)\} \\
&= tf(x_1) + (1-t)f(x_2).
\end{aligned}
$$

We next assume that $f(x_1) = 0$. If $f(x_2) = 0$, the target inequality is trivial. Suppose otherwise. Then,

$$
\begin{aligned}
&tf(x_1) + (1-t)f(x_2) \\
&= (1-t)f(x_2) \\
&= f((1-t)x_2) \\
&\leq f(tx_1 + (1-t)x_2).
\end{aligned}
$$

The last inequality is due to monotonicity. ///